$$Teorema:\ Completitud\ de\ \mathbb{R}$$

$$Una\ sucesión\ es\ convergente \Leftrightarrow es\ de\ Cauchy$$
$$(X_n)_n\ es\ convergente \Leftrightarrow (X_n)_n\ es\ de\ Cauchy$$

$$Demostración:$$

$1 \Rightarrow 2$

$$X_n\ es\ convergente \Rightarrow$$
$$\Rightarrow \forall \varepsilon > 0\ \ \exists n_0 \in \mathbb{N} / |x_n - x| < \frac{\varepsilon}{2} \quad si\ n \geq n_0$$

$$Sean\ \ n, m \geq n_0 \Rightarrow |x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
$$|x_m - x_n| < \varepsilon \Rightarrow X_n\ es\ de\ Cauchy$$

$2 \Rightarrow 1$

$A:$ *Toda sucesión de Cauchy está acotada* $\Rightarrow$ *al menos tiene un punto de acumulacón*

$$\forall \varepsilon = 1\ \ \exists n_0 \in \mathbb{N} / |x_m - x_n| < 1$$
$$sea\ \ n = n_0\ fijo \Rightarrow |x_m - x_{n_0}| < 1 \quad \forall m \geq n_0$$
$$|x_m| - |x_{n_0}| \leq |x_m - x_{n_0}| < 1 \Rightarrow |x_m| < 1 + |x_{n_0}| \quad \forall m \geq n_0$$
$$Sea\ K = máx\{|x_1|, |x_2|, |x_3|, \cdots, |x_{n_0 - 1}|, 1 + |x_{n_0}|\}$$

$$|x_m| \leq K \quad \forall m \in \mathbb{N}$$

$B:$ *Existe* $(X_{n_k}) \to x$ $\quad$ (*Teorema de Bolzano* $-$ *Weierstrass*)

$C:$ *Si* $(X_n)_n$ *es de Cauchy y existe* $(X_{n_k}) \to x \Rightarrow (X_n) \to x$

$$(X_n)\ es\ de\ Cauchy \Rightarrow \forall \varepsilon > 0\ \ \exists n_0 \in \mathbb{N} / |x_m - x_n| < \frac{\varepsilon}{2} \quad si\ n, m \geq n_0$$
$$(X_{n_k}) \to x \Rightarrow \exists k_0 \in \mathbb{N} / |x_{n_k} - x| < \frac{\varepsilon}{2} \quad si\ k \geq k_0$$

$$|x_n - x| = |x_n - x_{n_k} + x_{n_k} - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x|$$

$$Como\ (X_n)_n\ es\ de\ Cauchy \Rightarrow si\ n, n_k \geq n_0 \Rightarrow |x_n - x_{n_k}| < \frac{\varepsilon}{2}$$

$$Como\ (X_{n_k})\ es\ convergente \Rightarrow |x_{n_k} - x| < \frac{\varepsilon}{2} \quad si\ k \geq k_0$$

$$Sea\ n_1 = máx\{k_0, n_0\} \Rightarrow |x_n - x| < \varepsilon \quad \forall n \geq n_1$$